JYFL Preprint 18/1998

How to interpret black hole entropy?

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Abstract

We consider a possibility that the entropy of a Schwarzschild black hole has two different interpretations: The black hole entropy can be understood either as an outcome of a huge degeneracy in the mass eigenstates of the hole, or as a consequence of the fact that the interior region of black hole spacetime is separated from the exterior region by a horizon. In the latter case, no degeneracy in the mass eigenstates needs to be assumed. Our investigation is based on calculations performed with Lorentzian partition functions obtained for a whole maximally extended Schwarzschild spacetime, and for its right-hand-side exterior region. To check the correctness of our analysis we reproduce, in the leading order approximation, the Bekenstein–Hawking entropy of the Schwarzschild black hole.

Pacs: 04.60.Ds, 04.70.Dy

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I. INTRODUCTION

One of the most interesting branches of modern theoretical physics is black hole thermodynamics. The origin of this fascinating area of research can be traced back to the early 70's when it was observed that there are certain striking similarities between the laws of black hole mechanics and the laws of thermodynamics [1,2]. The similarities were mostly considered artificial until Hawking convincingly found – by taking into account quantum mechanical effects – that the exterior region of a black hole produces thermal radiation [3]. Ever since, the thermodynamical properties, like entropy, of black holes have been studied seriously, but many unsolved problems are still waiting for solutions. One of the most interesting issues is the question of the underlying microstates of the hole itself [4]. The unknown microstates determine the average values of the thermodynamical quantities of the hole, and it is very likely that the solution of the problem of the underlying microstates of the hole will give us valuable clues to the self-consistent quantum theory of gravity.

We all have been convinced by now by the fact that black holes bear entropy $S = \frac{1}{4}A$ [2]. This result originates from Bekenstein's and Hawking's work [1,3], but it has been reproduced by many authors since then [5–12]. The original calculation yielding the entropy was based on the semiclassical gravity, where spacetime was considered as a classical object, whereas matter fields were quantized in this classical but curved background spacetime. Some years later, Hawking was able to recover the same result by means of a Euclidean path-integral approach to quantum gravity [6]. Those approaches, however, failed to give an explanation to the black hole entropy at the fundamental level. More precisely, they did not provide a solution to the problem of underlying microstates of the hole: Since the black hole entropy is $\frac{1}{4}A$, one might expect that there are $\exp(\frac{1}{4}A)$ microstates corresponding to the same macrostate of the hole, and the problem is to identify these microstates. Search for the microstates has been going on for almost thirty years, and only recently the string-theoretical work of Strominger and Vafa has been able to give explicitly the number of the microstates [13]. In this paper, however, we shall investigate the black hole entropy by means of canonical methods.

The classical no-hair theorem states that after the collapse, when a black hole has settled down to a stationary state, its properties are determined by very few parameters observed far from the hole: These parameters are the mass M, the charge Q and the angular momentum $\vec{J}$ of the hole [14]. Thus, from the classical point of view, black holes have only three degrees of freedom. What has happened to the enormous amount of degrees of freedom of the collapsing matter? The no-hair theorem prompts one to believe that these degrees of freedom, and the information contained in them, is lost in the collapse, and that the entropy of a black hole may be understood as a measure of information loss during the gravitational collapse, because between the entropy and the information there is a well-known relationship given by Brillioin [15]: the decrease in information increases the entropy. This viewpoint is purely quantum-mechanical. According to quantum mechanics all the information from the collapsing star is not able to reach to an observer exterior to the newly formed event

horizon. In other words, all the microstates of the collapsing star cannot be measured by the external observer. This results to an increasing entropy S.

The question now arises: After the collapse of matter, are the degrees of freedom contained in the matter fields somehow encoded into the quantum states of the black hole spacetime itself, or have they vanished altogether, leaving no trace whatsoever? Of course, it is natural to claim that they are encoded into the quantum states of spacetime itself such that there is a vast $\exp(\frac{1}{4}A)$-fold degeneracy in the quantum states of the hole. This leads us to a conclusion that the total number of unknown quantum states of the black hole must be enormous, too. Thus, from a quantum-mechanical point of view, the number of the physical degrees of freedom of the hole is not limited to just few parameters. The contradiction between quantum and classical black holes is obvious: The number of physical degrees of freedom of the classical hole is three, whereas the number of physical degrees of freedom of the quantum black hole is enormous. The problem with this contradiction is that it is not quite clear how, starting from general relativity, quantization itself might bring along a huge number of additional degrees of freedom.

The purpose of this paper is to investigate a possibility that the entropy of a black hole is reproducable from the point of view of an external observer even if the observer takes into account the classical degrees of freedom only, and quantizes all the classically observed quantities, like mass, charge and angular momentum without assuming any degeneracy in the eigenstates of these quantities. In other words, we shall consider a possibility that the other but the classical degrees of freedom associated with the collapsing matter fields have vanished altogether. This point of view might provide a solution to the apparent contradiction related to the number of degrees of freedom of quantum and classical black holes. The key point in this paper is that we investigate the statistical mechanics of the *exterior* region of the black hole spacetime. This kind of a choice may be considered justified on grounds of the fact that the interior region of the black hole is separated from the exterior region by a horizon. Hence, an external observer cannot make any observations on the interior region, and one is justified to take a point of view that, for such an observer, physics of a black hole is physics of its exterior region. For the sake of convenience, we shall consider static and vacuum black holes only, but an analogous treatment could be performed for static electrovacuum black holes as well. The uniqueness theorem for nonrotating and vacuum black holes states that the Schwarzschild metric, with the mass parameter M, represents the only static and asymptotically flat black hole solution [14]. We shall see that the Bekenstein–Hawking entropy of a black hole is reproducable from the statistical mechanics of the exterior region of Schwarzschild black hole spacetime, even if we assume that there is no degeneracy in the mass eigenstates of the hole. We shall also see that the Bekenstein–Hawking entropy can be obtained for the whole spacetime as well, but in that case we must assume, *a priori*, an $\exp(\frac{1}{4}A)$-fold degeneracy in the mass eigenstates.

The analysis performed in this paper is based on the so called Hamiltonian thermodynamics of black holes. This branch of physics is an outgrowth of the analysis on the Hamiltonian

dynamics of the Schwarzschild spacetimes performed by Kuchař [16], and was initiated, among others, by Louko, Whiting, and Winters-Hilt [17,18]. We want to emphasize that the whole analysis in this paper is performed in Lorentzian spacetime without euclideanizing neither the Hamiltonian nor the action. The reason for performing the analysis in Lorentzian spacetime is that the interior of the Schwarzschild black hole is included in the analysis, too. In contrast, when one performs the euclideanization of the Schwarzschild spacetime action, the black hole interior is reduced to one point and thus it is somewhat questionable to talk about the quantum states of the hole. In our investigations the interior, as well as the exterior region of the hole plays an essential role.

This paper is organized in the following way: In Sec. II we describe very briefly the Hamiltonian formulation of Schwarzschild spacetimes and represent the Hamiltonian produced by Louko and Whiting [17] for the exterior region of the Schwarzschild black hole. In Sec. III we write two Lorentzian partition functions for the Schwarzschild black hole. The first of these partition functions describes the whole Kruskal spacetime, and the second the exterior region of the hole from the point of view of an observer at rest relative to the hole at the right-handed asymptotic infinity. These two partition functions appear to give identical partition functions for the radiaton emitted by the hole, if we use Bekenstein's proposal for a discrete area spectrum, and assume, in addition, that all the energy and the entropy of the hole is exactly converted into the energy and the entropy of the radiation.

The point we try to emphasize is that in order to obtain the partition function describing the whole spacetime, the observer must accept an $\exp(\frac{1}{4}A)$-fold degeneracy in the energy eigenstates of the hole, whereas no degeneracy needs to be assumed when one writes the partition function describing the exterior region of the hole. This will be the main result of this paper, and it has an interesting consequence: If one takes a view that, for an external observer, only the physical properties of the exterior region of the hole are relevant, then it is not necessary to consider the possible internal degrees of freedom of the hole itself, but it is sufficient to take into account only the classical physical degree of freedom of the Schwarzschild black hole, namely the mass M, to obtain the Bekenstein-Hawking entropy. This result is in harmony with the no-hair theorem and with the semiclassical results. Unless otherwise stated, we shall use natural units where $c = G = \hbar = k_B = 1$.

II. HAMILTONIAN THEORY

In this section we shall give a brief introduction to the classical Hamiltonian theory of spacetimes containing a Schwarzschild black hole. We have not aimed at a presentation that would give a technically detailed review on the subject; for more information, the authors recommend the reader to consult the papers written by Kuchař [16], and by Louko and Whiting [17]. The classical Hamiltonian theory presented in this section is based on those papers.

The first successful Hamiltonian formulation of general relativity was the so called ADM-

formalism, which was discovered by Arnowitt, Deser and Misner [19]. The basic idea of the ADM formalism is to foliate the spacetime manifold into the spacelike hypersurfaces where the time $t = constant$ and to use the components of the induced three-metric tensor q_{ab} as the coordinates of the configuration space. It is clear that the formalism depends heavily on the foliability of the spacetime manifold.

The ADM formalism of general relativity has four constraints per spacetime point, namely the Hamiltonian constraint and three diffeomorphism constraints. The three diffeomorphism constraints imply an invariance of general relativity under spacelike diffeomorphisms, and the remaining Hamiltonian constraint implies an invariance in time reparametrizations. In addition to these four constraints, the formalism has, of course, the Hamiltonian equations of motions. These equations plus the constraints of the Hamiltonian theory are equivalent to Einstein's field equations of general relativity.

When quantizing gravity canonically, we have to choose between two different possibilities: we either solve the constraints at the classical level, identify the physical degrees of freedom of the system and quantize the theory in the physical phase space, or we solve the quantum counterparts of the classical constraints. The former quantization method is known as the reduced phase space quantization, whereas the latter is known as the Dirac quantization [20]. In this paper we shall use the results based on the reduced phase space formalism. The quantization of the physical degrees of freedom of the system will not be performed explicitly. Quantum theories of the Schwarzschild black hole in the reduced phase space formalism have been constructed, among others, by Kuchař [16] and by Louko and Mäkelä [21].

The classical constraints for spherically symmetric, asymptotically flat vacuum spacetimes have been solved, among others, by Kuchař [16], and by Thiemann and Kastrup [22]. The only spherically symmetric, asymptotically flat vacuum solution to Einstein's field equations is the Schwarzschild solution. When the spacelike hypersurfaces, where $t = constant$, were chosen to go from the left to the right asymptotic infinities in the Kruskal diagram, crossing both the horizons, and the constraints were solved, Kuchař found that only two canonical degrees of freedom are left. If these two degrees of freedom are chosen to be the Schwarzschild mass m, and its conjugate momentum p_m, the classical action of the system is

$$S_{\mathrm{K}} = \int dt \, [p_m \dot{m} - m \, (N_+ + N_-)] \quad , \tag{2.1}$$

where N_+ and N_-, respectively, are the lapse functions at the right and at the left asymptotic infinities in the Kruskal diagram. The classical Hamiltonian of the whole maximally extended Schwarzschild black hole spacetime found by Kuchař can therefore be written in terms of the two physical phase space coordinates m and p_m as:

$$H_{\mathrm{whole}} = m \, (N_+ + N_-) \quad . \tag{2.2}$$

The classical Hamiltonian theory of the right-hand-side exterior region of the Schwarzschild black hole was investigated by Louko and Whiting [17]. It follows from the

analysis performed by those authors that, in the reduced phase space formalism, the classical Hamiltonian describing such a region of black hole spacetime can be written in terms of the Schwarzschild mass m and its conjugate momentum p_m as:

$$H_{\rm ext} = mN_{+} - \frac{1}{2}R_h^2 N_0 \quad , \tag{2.3}$$

where $R_h = 2m$ is the Schwarzschild radius, N_0 is a function of the global time t at the bifurcation two-sphere such that

$$\Theta := \int_{t_1}^{t_2} dt N_0(t) \tag{2.4}$$

is the boost parameter elapsed at the bifurcation two-sphere during the time interval $[t_1, t_2]$, and, as before, N_+ is the lapse function at the right-hand-side asymptotic infinity. We shall now give a brief review on the analysis performed by Louko and Whiting to produce the Hamiltonian (2.3).

Louko and Whiting considered a spacetime foliation where the spacelike hypersurfaces begin from the bifurcation two-sphere, and end at a right-hand-side timelike three-surface, i.e. at a "box wall" in the Kruskal diagram. With this choice, the spatial slices are entirely contained within the right-hand-side exterior region at the Kruskal spacetime. One of the main observations was that such foliations bring along an additional boundary term into the classical action. Hence, the Louko-Whiting boundary action $S_{\partial\Sigma}$ consists of terms resulting from the initial and the final spatial surfaces, that is, from the bifurcation two-sphere and from the "box wall". After solving the classical constraints, Louko and Whiting found that when the physical degrees of freedom are identified, the true Hamiltonian action is

$$S_{\rm LW} = \int dt \, (p_m \dot{m} - h(t)) \quad , \tag{2.5}$$

where $h(t)$ is the reduced Hamiltonian such that, when the radius of the initial boundary two-sphere does not change in time t, the Hamiltonian $h(t)$ is defined as

$$h(t) := \left(1 - \sqrt{1 - \frac{2m}{R}}\right) R\sqrt{-g_{tt}} - 2N_0(t)m^2 \quad , \tag{2.6}$$

where R is the time independent value of the radial coordinate of general spherically symmetric, asymptotically flat vacuum spacetime at the final timelike boundary i.e. at the "box wall", and g_{tt} is the $tt-$component of the metric tensor expressed as a function of the canonical variables after performing a canonical transformation, and of Lagrange's multipliers. Details can be seen in Ref. [17]. It is easy to see that if one transfers the "box wall" to the asymptotic infinity by taking the limit $R \to \infty$, the Hamiltonian $h(t)$ of Eq. (2.6) reduces to the Hamiltonian $H_{\rm ext}$ of Eq. (2.3).

III. HAMILTONIAN THERMODYNAMICS

If $\hat{H}$ is the Lorentzian Hamiltonian operator of a system, the partition function of the system is

$$Z = Tr \exp(-\beta \hat{H}) \ , \tag{3.1}$$

where $\beta = (k_B T)^{-1}$, k_B is Boltzmann's constant and T is the temperature of the system in a thermal equilibrium. The partition function (3.1) corresponds to the canonical ensemble and describes the thermodynamics of the system in a thermal equilibrium. Black holes can be considered as thermodynamical objects in a heat bath of temperature T [2,5,23,24]. Therefore, if the system under consideration is the whole maximally extended Schwarzschild spacetime, its Lorentzian Hamiltonian operator $\hat{H}$ would yield, via Eq. (3.1), a non-Euclideanized thermodynamical description of the whole black hole spacetime, and if the system under consideration is the exterior region of the Schwarzschild black hole only, the Lorentzian $\hat{H}$ would yield a non-Euclideanized partition function corresponding to the thermodynamical properties of the exterior region of the black hole spacetime. In practice, when one calculates the partition function (3.1) one needs to know, or assume, the density of the energy states of the system. We shall come to this crucial point later on this section.

We first obtain the partition function corresponding to the whole maximally extended Schwarzschild spacetime. Classically, H_{whole} may be understood as the total energy of the whole spacetime. To choose a specific observer, who measures the energy of the gravitational field, we fix the values of the lapse functions at asymptotic infinities. From the point of view of an observer at the right-hand-side infinity at rest with respect to the hole, we can set $N_- = 0$ and $N_+ = 1$. In other words, we have chosen the time coordinate at the right infinity to be the proper time of our observer and we have "frozen" the time evolution at the left infinity. The physical justification for such a choice is that our observer can make observations at just one asymptotic infinity. On the other hand, one may view the Schwarzschild mass m as the total energy of the Schwarzschild spacetime, measured by the distant observer. Hence, we may write $H_{\text{whole}} = m$.

To obtain the partition function for the Kruskal spacetime, we have to replace the operator $\hat{H}$ in Eq. (3.1) by an operator counterpart $\hat{H}_{\text{whole}}$ of the Hamiltonian H_{whole}. Hence, we get:

$$Z_{\text{whole}}(\beta) = Tr \exp(-\beta \hat{H}_{\text{whole}}) \quad . \tag{3.2}$$

During the recent years there has been increasing evidence that the mass spectrum of the black hole spacetime might be discrete [25]. If we denote these discrete mass eigenvalues of the mass operator $\hat{m} = \hat{H}_{\text{whole}}$ by m_n $(n = 0, 1, 2, \ldots)$ and the corresponding eigenvectors $|m_n\rangle$, we obtain an eigenvalue equation

$$\hat{H}_{\text{whole}} |m_n\rangle = \hat{m} |m_n\rangle = m_n |m_n\rangle \quad . \tag{3.3}$$

When the discrete energy spectrum is employed, the partition function (3.8) becomes

$$Z_{\text{whole}}(\beta) = \sum_{n=0}^{\infty} \langle m_n | \exp(-\beta \hat{m}) | m_n \rangle = \sum_{n=0}^{\infty} \exp(-\beta m_n) \quad . \tag{3.4}$$

Since the Bekenstein-Hawking entropy of black holes is

$$S_{\text{BH}} = \frac{1}{4} A \quad , \tag{3.5}$$

where A is the area of the event horizon, it is natural to assume an $\exp(\frac{1}{4}A)$-fold degeneracy in the possible mass eigenvalues m_n of the hole. This assumption of degeneracy is justified because entropy, in general, can be understood as a logarithm of the number of microstates corresponding to the same macrostate. Since for a Schwarzschild black hole with mass m, $A = 16\pi m^2$, we are prompted to define $g(m_n)$ as the number of degenerate states corresponding to the same mass eigenvalue m_n such that

$$\mathrm{g}(m_n) = \exp(4\pi m_n^2) \quad . \tag{3.6}$$

Hence, when the summation is performed over different mass eigenvalues only, we get for the whole maximally extended Schwarzschild spacetime the partition function which takes the following form:

$$\begin{aligned} Z_{\text{whole}}(\beta) &= \sum_{n=0}^{\infty} \mathrm{g}(m_n) \exp(-\beta m_n) \\ &= \sum_{n=0}^{\infty} \exp(-\beta m_n + 4\pi m_n^2) \quad . \end{aligned} \tag{3.7}$$

Before investigating the partition function (3.7) any further, let us, at this point, turn our attention to the partition function corresponding to the exterior region of the Schwarzschild black hole spacetime.

Classically, the Hamiltonian H_{ext} of the exterior region of the Schwarzschild black hole spacetime may be understood, in a certain foliation, as the total energy of the exterior region of the hole, although according to Bose et al. H_{ext} is the free energy of the whole black hole spacetime [26]. To obtain the corresponding partition function for the exterior region, we replace, as before, the operator $\hat{H}$ of Eq. (3.1) by an operator counterpart $\hat{H}_{\text{ext}}$ of H_{ext} and we require, as before, that the mass spectrum is discrete. In contrast to our discussion concerning the partition function of the whole spacetime, however, we assume the mass eigenstates to be non-degenerate. As a consequence, we get for the exterior region the partition function

$$\begin{aligned} Z_{\text{ext}}(\beta) &= \sum_{n=0}^{\infty} \Big\langle m_n \Big| \exp[-\beta(\hat{m} N_+ - 2\hat{m}^2 N_0)] \Big| m_n \Big\rangle \\ &= \sum_{n=0}^{\infty} \exp[-\beta(m_n N_+ - 2m_n^2 N_0)] \quad . \end{aligned} \tag{3.8}$$

Note that the partition function (3.8) is observer-dependent. To choose the same observer at the asymptotic infinity as in Eq. (3.7) we must, again, fix the value of the lapse function at the right-hand-side spatial infinity such that $N_+ \equiv 1$. From the point of view of such an observer, the partition function of the exterior region of the Schwarzschild black hole therefore takes the form:

$$Z_{\rm ext}(\beta) = \sum_{n=0}^{\infty} \exp[-\beta(m_n - 2m_n^2 N_0)] \quad . \tag{3.9}$$

To calculate the partition functions (3.7) and (3.9), we must assume, in addition, a specific spectrum for the mass eigenvalues m_n of the hole. In 1974 J. Bekenstein made a proposal, since then revived by several authors, that the possible eigenvalues of the area of the event horizon of the black hole are of the form [27]:

$$A_n = \gamma n l_{\rm Pl}^2 \quad , \tag{3.10}$$

where γ is a pure number of order one, n ranges over all non-negative integers, and $l_{\rm Pl} := (\hbar G/c^3)^{1/2}$ is the Planck length. When imposing this proposal, we find that the partition function of the whole Schwarzschild black hole spacetime is

$$Z_{\rm whole}(\beta) = \sum_{n=0}^{\infty} \exp\left(-\frac{\beta}{4}\sqrt{\frac{\gamma n}{\pi}} + \frac{\gamma n}{4}\right) \quad , \tag{3.11}$$

and the partition function of the exterior region of the Schwarzschild black hole is

$$Z_{\rm ext}(\beta) = \sum_{n=0}^{\infty} \exp\left[-\beta\left(\frac{1}{4}\sqrt{\frac{\gamma n}{\pi}} - \frac{N_0}{8}\frac{\gamma n}{\pi}\right)\right] \quad , \tag{3.12}$$

which both diverge very badly, indeed.

To actually calculate the partition functions (3.11) and (3.12), we have to deal with the problem of diverging partition functions. Kastrup has suggested some very original and interesting solutions to the divergency problem [12]. Our solution to the problem of a diverging partition function in the case of the whole maximally extended Schwarzschild spacetime is to study not the partition function of the whole spacetime itself but, instead, the partition function of the *radiation* emitted by the hole. When obtaining the partition function for the radiation, we assume that the evaporation of the hole is a reversible process. In other words, we assume that the entropy of the hole is converted exactly into the entropy of the radiation. A validity of this assumption has been investigated by Zurek [28]. His conclusion was that if the temperature of the heat bath is the same as that of the hole, then the black hole evaporation is a reversible process.

First, we choose the zero point of the energy emitted by the hole. This could be done in many ways, but we choose the total energy of the radiation emitted to be zero when the hole has evaporated completely leaving nothing but radiation. With this choice of the zero point of the total energy of the radiation, we find that the relationship between the energy

$E^{\rm rad}$ emitted by the hole and the mass m of the Schwarzschild black hole measured at the asymptotic right-hand-side infinity is

$$E^{\rm rad} = -m \quad . \tag{3.13}$$

If all the entropy of the hole is converted into the entropy of the radiation by means of transitions between degenerate black hole energy eigenstates, then the radiated energy spectrum is degenerate, too, and the number of the degenerate states corresponding to the same total energy emitted by the hole since its formation up to the point where the Schwarzschild mass has achieved the value m_n, is given by a function $g^{\rm rad}(m_n)$. It is fairly obvious that $g^{\rm rad}(m_n)$ increases when m_n decreases. In an ideal case, all the entropy of the hole is exactly converted to the entropy of the radiation. In that case we may choose

$$g^{\rm rad}(m_n) = \exp(\frac{1}{4}A_0 - 4\pi m_n^2) \quad , \tag{3.14}$$

where A_0 is the initial surface area of the black hole horizon, measured just before the hole has begun its evaporation. In other words, the decrease of the black hole entropy from $\frac{1}{4}A$ to $\frac{1}{4}(A-dA)$ increases the number of degenerate states of the radiation emitted by the hole by a factor $\exp(\frac{1}{4}dA)$. This choice reflects the fact that just after the hole has been formed, and not radiated yet, the entropy of the radiation is zero, whereas the entropy is $\frac{1}{4}A$ after the hole has evaporated completely.

Now, since $E^{\rm rad} = -m$ and $H_{\rm whole} = m$, we argue that

$$H^{\rm rad}_{\rm whole} = -m \quad . \tag{3.15}$$

To obtain the partition function for the radiation of the whole Schwarzschild spacetime, Eqs. (3.1), (3.3), (3.14) and (3.15) yield:

$$\begin{aligned} Z^{\rm rad}_{\rm whole}(\beta) &= \sum_{n=0}^{\infty} {\rm g}^{\rm rad}(m_n)\exp(\beta m_n) \\ &= \exp\left(\frac{1}{4}A_0\right)\sum_{n=0}^{\infty}\exp(\beta m_n - 4\pi m_n^2) \quad . \end{aligned} \tag{3.16}$$

When Bekenstein's proposal (3.10) is used, we get a partition function

$$Z^{\rm rad}_{\rm whole}(\beta) = \exp\left(\frac{1}{4}A_0\right)\sum_{n=0}^{\infty}\exp\left(\frac{\beta}{4}\sqrt{\frac{\gamma n}{\pi}} - \frac{\gamma n}{4}\right) \tag{3.17}$$

describing the radiation emitted by the Schwarzschild black hole. It is easy to see that $Z^{\rm rad}_{\rm whole}$ converges very nicely.

In comparison, let us obtain, by means of the same procedure as above, the partition function of the radiation emitted by the spacetime exterior to the Schwarzschild black hole. We choose the zero point of the energy of the radiation emitted by the external spacetime in the same way as before. This radiational energy should be understood as arising from

transitions between the energy states of the gravitational field corresponding to the exterior region of the Schwarzschild black hole spacetime. The zero points of energy emitted by both the Kruskal and the exterior spacetime can be chosen to coincide because the distant observer outside the hole observes the same energy $E^{\rm rad}$.

Now, since all the energy of the exterior region is assumed to be converted into the energy of the radiation, the Hamiltonian of the radiation of the exterior region may then be taken to be

$$H_{\rm ext}^{\rm rad} = -H_{\rm ext} \quad . \tag{3.18}$$

To obtain the partition function $Z_{\rm ext}^{\rm rad}$ one uses Eqs. (3.1), (3.3) and (3.18). These equations give a partition function

$$Z_{\rm ext}^{\rm rad}(\beta) = \exp\left(\frac{1}{4}A_0\right)\sum_{n=0}^{\infty}\exp(\beta m_n - 2N_0 m_n^2) \quad , \tag{3.19}$$

where we chose an appropriate normalization constant to the partition function. This is allowed, since the normalization does not have any effect on the measurable thermodynamical quantities, like temperature, of the system.

Applying, again, Bekenstein's proposal (3.10) to Eq. (3.19), we get

$$Z_{\rm ext}^{\rm rad}(\beta) = \exp\left(\frac{1}{4}A_0\right)\sum_{n=0}^{\infty}\exp\left[\frac{\beta}{4}\left(\sqrt{\frac{\gamma n}{\pi}} - \frac{N_0}{2}\frac{\gamma n}{\pi}\right)\right] \quad , \tag{3.20}$$

which, when keeping N_0 fixed, converges, too.

Let us next calculate the converging partition functions (3.17) and (3.20). Assuming that β is very big, we may approximate the sums (3.17) and (3.20) by integrals [29]:

$$\begin{aligned} Z_{\rm whole}^{\rm rad}(\beta) &\approx \exp\left(\frac{1}{4}A_0\right)\int_0^{\infty} dn\,\exp\left(\frac{\beta}{4}\sqrt{\frac{\gamma n}{\pi}} - \frac{\gamma n}{4}\right) \\ &= \exp\left(\frac{1}{4}A_0\right)\left[\frac{4}{\gamma} + \frac{\beta}{\gamma}\left[1 + {\rm erf}\left(\frac{\beta}{4\sqrt{\pi}}\right)\right]\exp\left(\frac{\beta^2}{16\pi}\right)\right] \quad , \end{aligned} \tag{3.21a}$$

$$\begin{aligned} Z_{\rm ext}^{\rm rad}(\beta) &\approx \exp\left(\frac{1}{4}A_0\right)\int_0^{\infty} dn\,\exp\left[\frac{\beta}{4}\left(\sqrt{\frac{\gamma n}{\pi}} - \frac{N_0}{2}\frac{\gamma n}{\pi}\right)\right] \\ &= \exp\left(\frac{1}{4}A_0\right)\left[\frac{8\pi}{\gamma\beta N_0} + \frac{4}{\gamma}\sqrt{\frac{2}{\beta}}\left(\frac{\pi}{N_0}\right)^{3/2}\left[\frac{1}{2} + \frac{1}{2}{\rm erf}\left(\frac{\beta}{8N_0}\right)^{1/2}\right]\exp\left(\frac{\beta}{8N_0}\right)\right] \quad , \end{aligned} \tag{3.21b}$$

where ${\rm erf}(x)$ is the error function.

If we, now, choose

$$N_0 = \frac{2\pi}{\beta} \quad , \tag{3.22}$$

then

$$Z^{\rm rad}_{\rm whole} = Z^{\rm rad}_{\rm ext} := Z^{\rm rad} \quad . \tag{3.23}$$

This is the main result of this paper. It should be noted that this result in not just an artefact of an approximation of a sum by an integral, but it holds even for exact expressions (3.17) and (3.20). We shall discuss the consequences of our result at the end of this section. Let us, in the meantime, try to justify Eq. (3.22).

If Eq. (3.22) holds, then Eqs. (3.21) give the semiclassical partition function of the radiation observed by an external observer at asymptotic infinity:

$$Z^{\rm rad}(\beta) \approx \exp\left(\frac{1}{4}A_0\right)\frac{2\beta}{\gamma}\exp\left(\frac{\beta^2}{16\pi}\right) \quad . \tag{3.24}$$

It is easy to show that the upper bound for the absolute error made, when replacing the sums (3.17) and (3.20) by integrals (3.21) is, in the leading order approximation, $\exp(1/4A_0 + \beta^2/16\pi)$. If one compares the result (3.24) to the absolute error made when replacing the sums by integrals, one notices that, for very big β, the fractional error is much smaller than unity. Hence, in the highest order approximation, the resulting partition function (3.24) approximates the sums (3.17) and (3.20) very well and, most importantly, the effect of the error bars on the thermodynamical quantities is negligibly small.

We now require that the energy expectation value of the radiation is:

$$\langle E^{\rm rad}\rangle := -\frac{\partial}{\partial\beta}\ln Z^{\rm rad}_{\rm ext}(\beta) = -\langle m\rangle \quad . \tag{3.25}$$

When β and $\langle m\rangle$ are taken to be very big, we get from (3.25):

$$-\frac{\beta}{8\pi} + \mathcal{O}(\beta^{-1}) = -\langle m\rangle \quad , \tag{3.26}$$

which, in turn, is the same as

$$\beta \approx 8\pi\langle m\rangle \quad . \tag{3.27}$$

This, on the other hand, corresponds to the choice

$$N_0 \approx \frac{1}{4\langle m\rangle} \quad . \tag{3.28}$$

It was noted by Bose et al. that when Einstein's field equations are satisfied, the quantity N_0 can be expressed as $N_0 = \kappa\frac{dT}{dt}$ [26], where T is the Schwarzschild time coordinate, i.e. the Killing time, t is the global time coordinate, and $\kappa = \frac{1}{4m}$ is the surface gravity of the black hole. Now, Eq. (3.28) implies that, in the semiclassical limit, $\frac{dT}{dt} = 1$, which states that the time coordinate t equals with the Schwarzschild time T. In other words, the meaning of the choice (3.22) is that the spacetime foliation near the horizon of the Schwarzschild black hole is determined by the Schwarzschild time coordinate T. Since the Schwarzschild time coordinate is just the time coordinate used by our external observer at

rest when he makes observations on the spacetime properties, one may regard the choice (3.22) justified on grounds of our aim to describe the black hole thermodynamics from the point of view of a faraway observer at rest. On the other hand, if one requires that, in the leading approximation, $N_0 \approx \frac{1}{4\langle m\rangle}$, and that $-\frac{\partial}{\partial\beta}\ln Z^{\rm rad}_{\rm ext}(\beta) = -\langle m\rangle$, then – as noted in Ref. [26] – one gets $\beta \approx 4C\langle m\rangle$, which gives $N_0 \approx \frac{C}{\beta}$, where the constant C can be chosen to be 2π. Hence, if we use a Schwarzschild -type foliation right from the beginning, we can obtain, up to a constant, the choice (3.22).

It is well known that the entropy S of any thermodynamical system, described by a partition function Z, can be calculated from an expression

$$S = \ln Z - \beta\frac{\partial}{\partial\beta}\ln Z \quad . \tag{3.29}$$

When substituting $Z^{\rm rad}$ into Eq. (3.29), one gets an approximation to the entropy of the black hole radiation:

$$S^{\rm rad} = \frac{1}{4}(A_0 - A) + \frac{1}{2}\ln A + \ln\left(\frac{4\sqrt{\pi}}{\gamma}\right) - 1 + \mathcal{O}\left(A^{-1/2}\right)\exp\left(-\frac{1}{4}A\right) \quad . \tag{3.30}$$

Hence, when the area of the black hole has shrinked from A_0 to A, the entropy carried away by the radiation is, in the leading order approximation, $\frac{1}{4}(A_0 - A)$. Under assumption that the black hole radiation is a reversible process, this result is compatible with the Bekenstein-Hawking expression for black hole entropy: A decrease of the area by an amount $A_0 - A$ decreases the entropy of the hole by an amount $\frac{1}{4}(A_0 - A)$. The error made when approximating the sum by an integral causes an error in the entropy which is of order $\mathcal{O}\left(A^{-1/2}\right)$.

We have obtained two partition functions $Z^{\rm rad}_{\rm whole}$ and $Z^{\rm rad}_{\rm ext}$. When obtaining the partition function $Z^{\rm rad}_{\rm ext}$ for the radiation emitted by the exterior region of the hole, the mass eigenstates were assumed to be discrete – as proposed by Bekenstein – and non-degenerate. When obtaining the partition function $Z^{\rm rad}_{\rm whole}$ of the radiation emitted by the whole Kruskal spacetime, however, we had to make an *ad hoc* assumption of an $\exp(\frac{1}{4}A)$-fold degeneracy in the discrete mass eigenstates of the hole to get the correct black hole entropy. Still, the two partition functions turned out to be exactly the same from the point of view of a distant observer. This is a very interesting result. Does it bear any implications relevant to the question of the nature of the black hole entropy?

Our investigation suggests two possible interpretations to the black hole entropy. The first interpretation is that the entropy of the hole is simply caused by the fact that an external observer cannot make any observations on the interior region of the black hole. As a consequence, the physics of a black hole is physics of its external region for such an observer, and it is sufficient to consider the statistical mechanics of that external region only. This interpretation is supported by our straightforward calculation which gives correctly the Bekenstein-Hawking entropy, without assuming any degeneracy in the mass eigenstates.

Another interpretation is more conservative: The entropy of the hole is interpreted as a huge degeneracy in the mass eigenstates of the whole black hole spacetime – including the interior region of the hole. When using this interpretation to obtain the Bekenstein-Hawking entropy one must make an *ad hoc* assumption about a vast $\exp(\frac{1}{4}A)$-fold degeneracy in the mass eigenstates.

What, then, are pro's and con's of the two viewpoints? From the first point of view, let us call it as an external point of view, the degrees of freedom of the collapsing matter, except the mass, are completely lost. Thus, the external point of view indicates that the information contained in the collapsing matter is not just hovering at some place, but completely and totally lost, whereas the conventional viewpoint somehow allows one to include the information about the degrees of freedom of the collapsing matter into the microstates of the hole itself. The loss of information, as known, leads to severe fundamental problems. These problems are discussed, for example, in [30–37]. On the other hand, the external view makes it possible to consider Schwarzschild black holes as objects having one physical degree of freedom only. This feature of the external point of view makes it appealing to us, as it – unlike the conventional point of view – is in perfect harmony with the no-hair theorem. Hence, one does not necessarily need to be concerned with how the quantization itself might bring along a vast number of additional degrees of freedom.

IV. CONCLUSION

In this paper we have obtained the partition function of the Schwarzschild black hole by means of two different Hamiltonians H_{whole} and H_{ext}. These Hamiltonians describe, respectively, the whole maximally extended Schwarzschild spacetime, and the exterior region of the Schwarzschild black hole. The whole Hamiltonian thermodynamics was considered in Lorentzian spacetime. The main reason for not producing a euclideanized partition function of the Schwarzschild black hole was that we wanted to include the interior of the black hole in the analysis. After writing the Hamiltonians, we obtained the corresponding partition functions which can be viewed, respectively, as the partition functions of the whole maximally extended Schwarzschild spacetime, and the spacetime region exterior to the black hole, from the point of view of a faraway observer at rest.

We found that these two partition functions coincide. To obtain this result, however, we were compelled to assume an $\exp(\frac{1}{4}A)$-fold degeneracy in the mass eigenstates when calculating the partition function of the whole spacetime, whereas no degeneracy was needed to be assumed when calculating the partition function for the exterior region. In addition, we chose the spacetime foliation near the horizon of the Schwarzschild black hole to be determined by the Schwarzschild time coordinate T which fixed, up to a constant, the quantity N_0.

To check the correctness of our partition functions, we used Bekenstein's proposal for a discrete area spectrum of black holes to calculate the Bekenstein-Hawking entropy. Un-

fortunately, the partition functions of the whole black hole spacetime, and the spacetime region exterior to the hole were found to diverge, but we managed to solve the divergency problem, however, by turning our attention to the *radiation* emitted by the hole. More precisely, we obtained the partition functions of radiation emitted when either the whole black hole spacetime or its exterior region are assumed to perform transitions from a one state to another. When obtaining the partition functions of radiation we assumed that the evaporation of the hole is a reversible process and that all the energy and the entropy of the hole are exactly converted to the energy and the entropy of radiation. The resulting partition functions for radiation were found to converge very nicely producing, in the leading order approximation, the Bekenstein-Hawking entropy of black holes.

Our investigation suggested that the black hole entropy can be interpreted in two possible ways. First, there is the conservative view that the entropy of black holes may be understood as a result of a huge degeneracy in the mass eigenstates of the whole black hole spacetime. The degeneracy of the eigenstates might somehow, in a still unexplained manner, allow one to include the degrees of freedom of the collapsed matter, but the view is in contradiction with the no-hair theorem. The second view – called the external point of view – is that the entropy of black holes is, quite simply, caused by the fact that the interior region of black hole spacetime is separated from its exterior region by a horizon. Because of that, one might be justified to take a view that black hole statistical mechanics is, for an external observer, statistical mechanics of its exterior region. This point of view allows one to obtain the Bekenstein-Hawking entropy without assuming any degeneracy in the mass eigenstates of the hole. The result is in harmony with the no-hair theorem, but allows a complete loss of information, since the degrees of freedom of the matter, except the total mass M, have vanished. We have thus two complementary points of view to the interpretation of black hole entropy, of which neither is quite completely satisfactory: The conservative view is in conflict with the no-hair theorem, whereas the external point of view, although it is physically appealing and in harmony with the no-hair theorem, implies a tremendous loss of information. It remains to be seen whether these two possible interpretations could somehow be unified into a single, consistent description of black holes.

ACKNOWLEDGMENTS

We are grateful to Jorma Louko and Markku Lehto for their constructive criticism during the preparation of this paper. P. R. was supported by the Finnish Cultural Foundation, Wihuri Foundation and Nyyssönen Foundation.

REFERENCES

[1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[2] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys., **31**, 161 (1973).
[3] S. W. Hawking, Commun. Math. Phys., **43**, 199 (1975).
[4] See, for example, the articles written by R. D. Sorkin and G. T. Horowitz, in *Black Holes and Relativistic Stars*, edited by R. M. Wald (University Chicago Press, Chicago, 1998).
[5] J. B. Hartle and S. W. Hawking, Phys. Rev. D **13**, 2188 (1976).
[6] S. W. Hawking, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979)
[7] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).
[8] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1420 (1993).
[9] S. W. Hawking, G. T. Horowitz and S. F. Ross, Phys. Rev. D **51**, 4302 (1995).
[10] C. Teitelboim, Phys. Rev. D **51**, 4315 (1995).
[11] E. A. Martinez, Phys. Rev. D **52**, 5732 (1995).
[12] H. A. Kastrup, Phys. Lett. B **419**, 40 (1998), Phys. Lett. B **413**, 267 (1997), hep-th/9803180.
[13] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[14] See, for example, M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, 1996).
[15] L. Brillouin, *Science and Information Theory* (Academic, New York, 1956).
[16] K. V. Kuchař, Phys. Rev. D **50**, 3961 (1994).
[17] J. Louko and B. F. Whiting, Phys. Rev. D **51**, 5583 (1995).
[18] J. Louko and S. N. Winters-Hilt, Phys. Rev. D **54**, 2647 (1996).
[19] R. Arnowitt, S. Deser and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[20] C. J. Isham, gr-qc/9210011.
[21] J. Louko and J. Mäkelä, Phys. Rev. D **54**, 4982 (1996).
[22] T. Thiemann and H. A. Kastrup, Nucl. Phys. **B399**, 211 (1993).
[23] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[24] W. Israel, Phys. Lett. A **57**, 107 (1976).
[25] See references quoted by J. D. Bekenstein, gr-qc/9808028 and gr-qc/9710076.
[26] S. Bose, L. Parker and Y. Peleg, Phys. Rev. D. **56**, 987 (1997).
[27] J. D. Bekenstein, Lett. Nuovo Cimento **11**, 467 (1974).
[28] W. H. Zurek, Phys. Rev. Lett. **49**, 1683 (1982).
[29] M. R. Spiegel, *Schaum's Outline Series: Mathematical Handbook of Formulas and Tables* (McGraw-Hill, 1992).
[30] S. W. Hawking, Phys. Rev. D **14**, 2460 (1976).
[31] G. 't Hooft, Nucl. Phys. **B335**, 138 (1990).
[32] J. Preskill, hep-th/9209058.

[33] J. A. Harvey and A. Strominger, hep-th/9209055.
[34] T. Banks, M. O'Loughlin and A. Strominger, hep-th/9211030.
[35] D. N. Page, hep-th/9306083.
[36] U. F. Danielsson and M. Schiffer, Phys. Rev. D **48**, 4779 (1993).
[37] S. B. Giddings, hep-th/9508151.